Hypothesis Testing

STA 721: Lecture 13

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Outline

Hypothesis Testing:

- The hypothesis of no effects
  - F-tests
  - Null distribution
  - Decision procedure
- Testing submodels
  - Extra sum of squares

Readings:

- Christensen Appendix C, Chapter 3

The Hypothesis of No Effects

Suppose we believe the model

$$\mathrm{M1} \qquad \mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon} \qquad \boldsymbol{\epsilon} \sim \mathsf{N}(0, \sigma^2 \mathbf{I}_n)$$

but hypothesize that there is no effect of the $\mathbf{X}$ variables on $\mathbf{Y}$

- If this were true, then the distribution for $\mathbf{Y}$ would be

$$\mathrm{M0} \qquad \mathbf{Y} = \boldsymbol{\epsilon} \qquad \boldsymbol{\epsilon} \sim \mathsf{N}(0, \sigma^2 \mathbf{I}_n)$$

- For M1, the distribution of $\mathbf{Y}$ is a collection of normal distributions with $\boldsymbol{\mu} \in C(\mathbf{X})$ and Covariance a scalar multiple of the $\mathbf{I}$
- the distributions for the data $\mathbf{Y}$ under M0 is a subset of the distributions under M1 or *submodel* of M1 with $\boldsymbol{\mu} = \mathbf{0}$
- Observations $\mathbf{Y}$ may give us evidence that supports or rejects our hypothesis that the null model, M0, is true

Goal

Our goals are to

- obtain a numerical summary of the evidence
- come up with a decision-making procedure that decides between M1 and M0,
- (frequentist) control the probability of making a certain type of incorrect decision

Procedure based on the following steps:

1. Test statistic: compute a statistic $t(\mathbf{Y}, \mathbf{X})$, a function of observable data;
2. Null distribution: compare $t(\mathbf{Y}, \mathbf{X})$ to the types of values we would expect if M0 is true
3. Decision rule: accept M0 if $t(\mathbf{Y}, \mathbf{X})$ is in accord with its distribution under M0, otherwise reject the submodel M0

Intuition

If $\hat{\boldsymbol{\beta}} \approx \boldsymbol{\beta}$ then

- if $\boldsymbol{\beta} = \mathbf{0}$, then $\mathbf{X}\hat{\boldsymbol{\beta}} \approx \mathbf{0}$
- if $\boldsymbol{\beta} \neq \mathbf{0}$, then $\mathbf{X}\hat{\boldsymbol{\beta}} \not\approx \mathbf{0}$
- If the null model M0 is correct, then $\|\mathbf{X}\hat{\boldsymbol{\beta}}\|^2$ should be small
- If incorrect, $\|\mathbf{X}\hat{\boldsymbol{\beta}}\|^2$ should be big
- We need to quantify this intuition

Decomposition

$$
\begin{aligned}
\mathbf{X}\hat{\boldsymbol{\beta}} &= \mathbf{P}_{\mathbf{X}}\mathbf{Y} \\
&= \mathbf{X}\boldsymbol{\beta} + \mathbf{P}\boldsymbol{\epsilon}
\end{aligned}
$$

$$
\begin{aligned}
\|\mathbf{X}\hat{\boldsymbol{\beta}}\|^2 &= (\mathbf{X}\boldsymbol{\beta} + \mathbf{P}\boldsymbol{\epsilon})^T(\mathbf{X}\boldsymbol{\beta} + \mathbf{P}\boldsymbol{\epsilon}) \\
&= \boldsymbol{\beta}^T\mathbf{X}^T\mathbf{X}\boldsymbol{\beta} + 2\boldsymbol{\beta}^T\mathbf{X}^T\mathbf{P}\boldsymbol{\epsilon} + \boldsymbol{\epsilon}^T\mathbf{P}\boldsymbol{\epsilon} \\
&= \|\mathbf{X}\boldsymbol{\beta}\|^2 + 2\boldsymbol{\beta}^T\mathbf{X}^T\boldsymbol{\epsilon} + \boldsymbol{\epsilon}^T\mathbf{P}\boldsymbol{\epsilon}
\end{aligned}
$$

How big is $\|\mathbf{X}\hat{\boldsymbol{\beta}}\|^2$ on average? How big do we expect it to be under our two models?

Take expectations:

$$
\begin{aligned}
\textsf{E}[\|\mathbf{X}\hat{\boldsymbol{\beta}}\|^2] &= \|\mathbf{X}\boldsymbol{\beta}\|^2 + \textsf{E}[2\boldsymbol{\beta}^T\mathbf{X}^T\boldsymbol{\epsilon}] + \textsf{E}[\boldsymbol{\epsilon}^T\mathbf{P}\boldsymbol{\epsilon}] \\
&= \|\mathbf{X}\boldsymbol{\beta}\|^2 + 0 + \sigma^2\textsf{tr}(\mathbf{P}) = \|\mathbf{X}\boldsymbol{\beta}\|^2 + \sigma^2 p
\end{aligned}
$$

- if $\boldsymbol{\beta} = \mathbf{0}$, then $\textsf{E}[\|\mathbf{X}\hat{\boldsymbol{\beta}}\|^2] = \sigma^2 p$

Comparison

If we knew σ^2, then

- if $\|\mathbf{X}\hat{\boldsymbol{\beta}}\|^2/p \approx \sigma^2$, we might decide M0 would be reasonable
- if $\|\mathbf{X}\hat{\boldsymbol{\beta}}\|^2/p \gg \sigma^2$, then we might decide M0 is unreasonable

But we do not know σ^2

- if we estimate σ^2 by $s^2 = \frac{\mathbf{Y}^T(\mathbf{I}-\mathbf{P})\mathbf{Y}}{n-p}$, then
  - if $\|\mathbf{X}\hat{\boldsymbol{\beta}}\|^2/p \approx s^2$, we might decide M0 would be reasonable
  - if $\|\mathbf{X}\hat{\boldsymbol{\beta}}\|^2/p \gg s^2$, then we might decide M0 is unreasonable

Test Statistic

Note: if the null model M0 is correct ($\boldsymbol{\beta} = \mathbf{0}$), then **both**

- $\|\mathbf{X}\hat{\boldsymbol{\beta}}\|/p$
- SSE$/(n-p) = \frac{\mathbf{Y}^T(\mathbf{I}-\mathbf{P})\mathbf{Y}}{n-p}$

are unbiased estimates of σ^2

If the null model is not correct, but the linear model M1 is correct, then

Distributions under the Null Model M0

- SSE $\sim \sigma^2 \chi^2_{n-p}$
- $\|\mathbf{X}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})\|^2 \sim \sigma^2 \chi^2_p$

so under the null model M0 ($\boldsymbol{\beta} = \mathbf{0}$), we have

- $\mathsf{SSE} \sim \sigma^2 \chi^2_{n-p}$
- $\|\mathbf{X}\hat{\boldsymbol{\beta}}\|^2 \sim \sigma^2 \chi^2_p$
- they are statistically independent (why?)
- so the ratio

$$
\begin{aligned}
t(\mathbf{Y}, \mathbf{X}) &= \frac{\mathsf{RSS}/p}{\mathsf{SSE}/(n-p)} = \frac{(\mathsf{RSS}/\sigma^2)/p}{(\mathsf{SSE}/\sigma^2)/(n-p)} \\
&\stackrel{\mathrm{D}}{=} \frac{\chi^2_p/p}{\chi^2_{n-p}/(n-p)} \qquad \text{is independent of } \sigma^2
\end{aligned}
$$

is independent of σ^2

F Distribution

▼ **Definition:** F distribution

If $X_1 \sim \chi^2_{d1}$ and $X_2 \sim \chi^2_{d2}$ and are independent, then the ratio

$$F = \frac{X_1/d1}{X_2/d2}$$

has an $F_{d1,d2}$ distribution with $d1$ and $d2$ degrees of freedom.

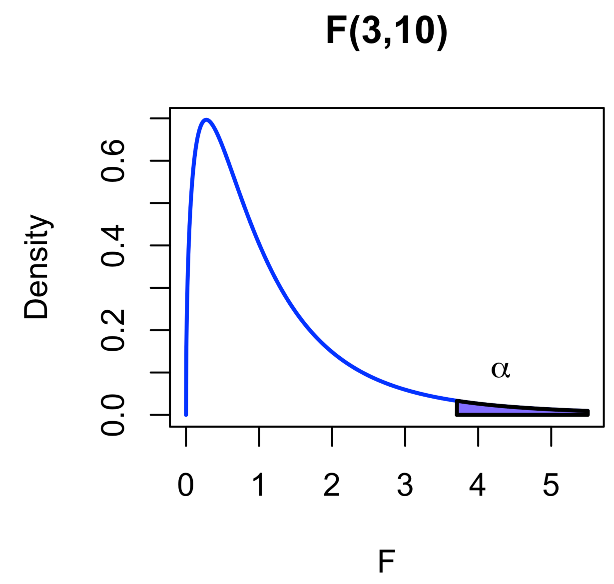
- $F(\mathbf{Y}) \equiv t(\mathbf{Y}, \mathbf{X}) = \frac{\textsf{RSS}/p}{\textsf{SSE}/(n-p)}$ has an $F_{p,n-p}$ distribution under the null model M0

Decision Procedure

We will accept M0 that $\boldsymbol{\beta} = \mathbf{0}$ unless $F(\mathbf{Y})$ is large compared to an $F_{p,n-p}$ distribution.

- accept M0: $\boldsymbol{\beta} = \mathbf{0}$ if $F(\mathbf{Y}) < F_{p,n-p,1-\alpha}$
- $F_{p,n-p,1-\alpha}$ is the $1-\alpha$ quantile of a $F_{p,n-p}$
- reject M0: $\boldsymbol{\beta} = \mathbf{0}$ if $F(\mathbf{Y}) > F_{p,n-p,1-\alpha}$
- the probability that we reject M0 when it is true, is

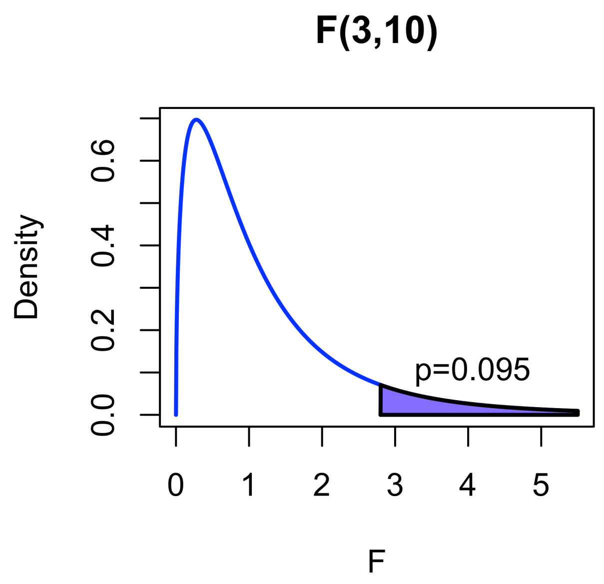
$$
\begin{aligned}
\Pr(\text{ reject M0} \mid \text{ M0 true}) &= \Pr(F(\mathbf{Y}) > F_{p,n-p,1-\alpha} \mid \boldsymbol{\beta} = \mathbf{0}) \\
&= \alpha
\end{aligned}
$$



P-values

Instead of just declaring that M0 is true or false, statistical analyses report how extreme $F(\mathbf{Y})$ is compared to its null distribution.

- This is usually reported in terms of the p-value:
  - the value $p \in (0, 1)$ such that $F(\mathbf{Y})$ is the $(1-p)$ quantile of the $F_{p,n-p}$ distribution
  - the probability that a random variable $F \sim F_{p,n-p}$ is larger than the observed value $F(\mathbf{Y})$, if the null model is true
- it is not the $\Pr(\text{M0 is true})$ based on the observed data



Testing SubModels

We are usually not interested in testing that all of the coefficients are zero if there is an intercept in the model

- But we can use the same idea to test submodels
- We assume the Gaussian Linear Model

$$M1 \quad \mathbf{Y} \sim \mathsf{N}(\mathbf{W}\boldsymbol{\alpha} + \mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I}) \equiv \mathsf{N}(\mathbf{Z}\boldsymbol{\theta}, \sigma^2\mathbf{I})$$

  where $\mathbf{W}$ is $n \times q$, $\mathbf{X}$ is $n \times p$, $\mathbf{Z} = [\mathbf{W}\mathbf{X}]$,
- We wish to evaluate the hypothesis $\boldsymbol{\beta} = \mathbf{0}$
- equivalent to comparing M1 to M0:

$$M0 \quad \mathbf{Y} \sim \mathsf{N}(\mathbf{W}\boldsymbol{\alpha}, \sigma^2\mathbf{I})$$

Intuition

Devise a test statistic and procedure by

- fitting the full model M1 $\mathbf{Y} \sim \mathsf{N}(\mathbf{W}\boldsymbol{\alpha} + \mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I})$
- fitting the reduced/null model M0 $\mathbf{Y} \sim \mathsf{N}(\mathbf{W}\boldsymbol{\alpha}, \sigma^2\mathbf{I})$
- accept M0 if the null model fits about as well as the full model
- reject M0 if the null model fits much worse than the full model
- measure fit through SSE_{M0} and SSE_{M1}

$$
\begin{aligned}
\mathsf{SSE}_{M1} &= \min_{\boldsymbol{\alpha},\boldsymbol{\beta}} \|\mathbf{Y} - (\mathbf{W}\boldsymbol{\alpha} + \mathbf{X}\boldsymbol{\beta})\|^2 \\
&= \min_{\boldsymbol{\theta}} \|\mathbf{Y} - \mathbf{Z}\boldsymbol{\theta}\|^2 = \mathbf{Y}^T(\mathbf{I} - \mathbf{P}_{\mathbf{Z}})\mathbf{Y} \\
\mathsf{SSE}_{M0} &= \min_{\boldsymbol{\alpha}} \|\mathbf{Y} - \mathbf{W}\boldsymbol{\alpha}\|^2 = \mathbf{Y}^T(\mathbf{I} - \mathbf{P}_{\mathbf{W}})\mathbf{Y}
\end{aligned}
$$

Extra Sum of Squares

Approach 1: accept/choose the null model if $\textsf{SSE}_{M0} < \textsf{SSE}_{M1}$, and choose the full model if $\textsf{SSE}_{M1} < \textsf{SSE}_{M0}$.

- but $\textsf{SSE}_{M1}$ is always less than $\textsf{SSE}_{M0}$

Approach 2: instead reject M1 $\boldsymbol{\beta} = \mathbf{0}$ if $\textsf{SSE}_{M0}$ is much bigger than $\textsf{SSE}_{M1}$.

- Specifically, reject M1 $\boldsymbol{\beta} = \mathbf{0}$ if $\textsf{SSE}_{M0} - \textsf{SSE}_{M1}$ is much bigger than what we would expect if the null hypothesis M0 were true.

Need:

- the null distribution of $\textsf{SSE}_{M0}$
- the null distribution of $\textsf{SSE}_{M1}$
- the null distribution of their difference $\textsf{SSE}_{M0} - \textsf{SSE}_{M1}$

Distributions

Distribution under the full model M1

$$\mathsf{SSE}_{M1} = \mathbf{Y}^T(\mathbf{I} - \mathbf{P_Z})\mathbf{Y} \sim \sigma^2\chi^2_{n-q-p}$$

- true whether or not $\boldsymbol{\beta} = \mathbf{0}$
- $\mathsf{E}[\mathsf{SSE}_{M1}] = E[\mathbf{Y}^T(\mathbf{I} - \mathbf{P_Z})\mathbf{Y}] = \sigma^2(n - q - p)$

Distribution under the null model M0

$$\mathsf{SSE}_{M0} = \mathbf{Y}^T(\mathbf{I} - \mathbf{P_W})\mathbf{Y} \sim \sigma^2\chi^2_{n-q}$$

- true if $\boldsymbol{\beta} = \mathbf{0}$
- $\mathsf{E}[\mathsf{SSE}_{M0}] = E[\mathbf{Y}^T(\mathbf{I} - \mathbf{P_W})\mathbf{Y}] = \sigma^2(n - q)$
- if $\boldsymbol{\beta} \neq \mathbf{0}$ then SSE_{M0} has a non-central χ^2_{n-q} distribution

Expected Value of SSE$_{M0}$ under M1

- Rewrite $(\mathbf{I} - \mathbf{P_W})\mathbf{Y}$ under M1:

$$\begin{aligned}(\mathbf{I} - \mathbf{P_W})\mathbf{Y} &= (\mathbf{I} - \mathbf{P_W})(\mathbf{W}\boldsymbol{\alpha} + \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}) \\ &= (\mathbf{I} - \mathbf{P_W})\mathbf{X}\boldsymbol{\beta} + (\mathbf{I} - \mathbf{P_W})\boldsymbol{\epsilon}\end{aligned}$$

- compute $\textsf{E}[\textsf{SSE}_{M0}]$ under M1:

$$\begin{aligned}\textsf{E}[\mathbf{Y}^T(\mathbf{I} - \mathbf{P_W})\mathbf{Y}] &= \boldsymbol{\beta}^T\mathbf{X}^T(\mathbf{I} - \mathbf{P_W})\mathbf{X}\boldsymbol{\beta} + \textsf{E}[\boldsymbol{\epsilon}^T(\mathbf{I} - \mathbf{P_W})\boldsymbol{\epsilon}] \\ &= \boldsymbol{\beta}^T\mathbf{X}^T(\mathbf{I} - \mathbf{P_W})\mathbf{X}\boldsymbol{\beta} + \sigma^2 \textsf{tr}(\mathbf{I} - \mathbf{P_W}) \\ &= \boldsymbol{\beta}^T\mathbf{X}^T(\mathbf{I} - \mathbf{P_W})\mathbf{X}\boldsymbol{\beta} + \sigma^2(n - q)\end{aligned}$$

- under M0, both $\textsf{SSE}_{M0}/(n-q)$ and $\textsf{SSE}_{M1}/(n-q-p)$ are unbiased estimates of σ^2
- but does the ratio $\frac{\textsf{SSE}_{M0}/(n-q)}{\textsf{SSE}_{M1}/(n-q-p)}$ have a F distribution?

Extra Sum of Squares

Rewrite $\textsf{SSE}_{M0}$:

$$\begin{aligned}
\textsf{SSE}_{M0} &= \mathbf{Y}^T(\mathbf{I} - \mathbf{P_W})\mathbf{Y} \\
&= \mathbf{Y}^T(\mathbf{I} - \mathbf{P_Z} + \mathbf{P_Z} - \mathbf{P_W})\mathbf{Y} \\
&= \mathbf{Y}^T(\mathbf{I} - \mathbf{P_Z})\mathbf{Y} + \mathbf{Y}^T(\mathbf{P_Z} - \mathbf{P_W})\mathbf{Y} \\
&= \textsf{SSE}_{M1} + \mathbf{Y}^T(\mathbf{P_Z} - \mathbf{P_W})\mathbf{Y}
\end{aligned}$$

Extra Sum of Squares:

$$\textsf{SSE}_{M0} - \textsf{SSE}_{M1} = \mathbf{Y}^T(\mathbf{P_Z} - \mathbf{P_W})\mathbf{Y}$$

- is $\mathbf{P_Z} - \mathbf{P_W}$ is a projection matrix?
- onto what space? along what space ?
- what is the distribution of $\textsf{SSE}_{M0} - \textsf{SSE}_{M1}$ under the null model M0? under M1?
- is it independent of $\textsf{SSE}_{M1}$?