Hypothesis Testing Related to SubModels

STA 721: Lecture 14

Merlise Clyde (clyde@duke.edu)
Duke University

Outline

Hypothesis Testing:

- Testing submodels
  - Extra sum of squares
  - F-tests
  - Null distribution
  - Decision procedure
  - P-values
- Testing individual coefficients
  - t-tests
- Likelihood Ratio Tests

Readings:

- Christensen Appendix C, Chapter 3

Testing Recap

- We assume the Gaussian Linear Model

$$\mathrm{M1} \quad \mathbf{Y} \sim \mathsf{N}(\mathbf{W}\boldsymbol{\alpha} + \mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I}) \equiv \mathsf{N}(\mathbf{Z}\boldsymbol{\theta}, \sigma^2\mathbf{I})$$

  where $\mathbf{W}$ is $n \times q$, $\mathbf{X}$ is $n \times p$, $\mathbf{Z} = [\mathbf{W}\mathbf{X}]$,
- We wish to evaluate the hypothesis $\boldsymbol{\beta} = \mathbf{0}$
- equivalent to comparing M1 to M0:

$$\mathrm{M0} \quad \mathbf{Y} \sim \mathsf{N}(\mathbf{W}\boldsymbol{\alpha}, \sigma^2\mathbf{I})$$

- $\textsf{SSE}_{M0}/(n-q)$ and $\textsf{SSE}_{M1}/(n-q-p)$ are unbiased estimates of σ^2 under null model M0
- but the ratio $\frac{\textsf{SSE}_{M0}/(n-q)}{\textsf{SSE}_{M1}/(n-q-p)}$ does not have a F distribution

Extra Sum of Squares

Rewrite $\textsf{SSE}_{M0}$:

$$\begin{aligned}\textsf{SSE}_{M0} &= \mathbf{Y}^T(\mathbf{I} - \mathbf{P_W})\mathbf{Y} \\ &= \mathbf{Y}^T(\mathbf{I} - \mathbf{P_Z} + \mathbf{P_Z} - \mathbf{P_W})\mathbf{Y} \\ &= \mathbf{Y}^T(\mathbf{I} - \mathbf{P_Z})\mathbf{Y} + \mathbf{Y}^T(\mathbf{P_Z} - \mathbf{P_W})\mathbf{Y} \\ &= \textsf{SSE}_{M1} + \mathbf{Y}^T(\mathbf{P_Z} - \mathbf{P_W})\mathbf{Y}\end{aligned}$$

Extra Sum of Squares:

$$\textsf{SSE}_{M0} - \textsf{SSE}_{M1} = \mathbf{Y}^T(\mathbf{P_Z} - \mathbf{P_W})\mathbf{Y}$$

Expectation of Extra Sum of Squares

$$\textsf{E}[\textsf{SSE}_{M0} - \textsf{SSE}_{M1}] = \textsf{E}[\mathbf{Y}^T(\mathbf{P_Z} - \mathbf{P_W})\mathbf{Y}]$$

- under M0: $\boldsymbol{\mu} = \mathbf{W}\boldsymbol{\alpha}$

$$\begin{aligned} \textsf{E}[(\mathbf{P_Z} - \mathbf{P_W})\mathbf{Y}] &= \mathbf{P_Z}\mathbf{W}\boldsymbol{\alpha} - \mathbf{P_W}\mathbf{W}\boldsymbol{\alpha} \\ &= \mathbf{W}\boldsymbol{\alpha}\mathbf{W}_{\boldsymbol{\alpha}} \\ &= \mathbf{0} \\ \textsf{E}[\mathbf{Y}^T(\mathbf{P_Z} - \mathbf{P_W})\mathbf{Y}] &= \sigma^2(\textsf{tr}\mathbf{P_Z} + \textsf{tr}\mathbf{P_W}) \\ &= \sigma^2(q + p - q) = p\sigma^2 \end{aligned}$$

- under M1: $\boldsymbol{\mu} = \mathbf{X}\boldsymbol{\beta} + \mathbf{W}\boldsymbol{\alpha}$

$$\begin{aligned} \textsf{E}[(\mathbf{P_Z} - \mathbf{P_W})\mathbf{Y}] &= \mathbf{X}\boldsymbol{\beta} + \mathbf{W}\boldsymbol{\alpha} - \mathbf{P_W}\mathbf{X}\boldsymbol{\beta} - \mathbf{W}\boldsymbol{\alpha} \\ &= (\mathbf{I} - \mathbf{P_W})\mathbf{X}\boldsymbol{\beta} \\ \textsf{E}[\mathbf{Y}^T(\mathbf{P_Z} - \mathbf{P_W})\mathbf{Y}] &= p\sigma^2 + \boldsymbol{\beta}^T\mathbf{X}^T(\mathbf{I} - \mathbf{P_W})\mathbf{X}\boldsymbol{\beta} \end{aligned}$$

Test Statistic

Propose ratio:

$$F = \frac{(\textsf{SSE}_{M0} - \textsf{SSE}_{M1})/p}{\textsf{SSE}_{M1}/(n-q-p)} = \frac{\mathbf{Y}^T(\mathbf{P_Z} - \mathbf{P_W})\mathbf{Y}/p}{\textsf{SSE}_{M1}/(n-q-p)}$$

as a test statistic.

Does F have an F distribution under M0?

- denominator $\textsf{SSE}_{M1}/\sigma^2$ does have a χ^2 distribution?
- does numerator $\textsf{SSE}_{M0}/\sigma^2$ have a χ^2 distribution?
- are they independent?

Properties of $\mathbf{P_Z} - \mathbf{P_W}$

To show that $\mathbf{Y}^T(\mathbf{P_Z} - \mathbf{P_W})\mathbf{Y}$ has a χ^2 distribution under M0 or M1, we need to show that $\mathbf{P_Z} - \mathbf{P_W}$ is a projection matrix.

- symmetric?
- idempotent?

$$
\begin{aligned}
(\mathbf{P_Z} - \mathbf{P_W})^2 &= \mathbf{P}_\mathbf{Z}^2 - \mathbf{P_Z}\mathbf{P_W} - \mathbf{P_W}\mathbf{P_Z} + \mathbf{P}_\mathbf{W}^2 \\
&= \mathbf{P_Z} - \mathbf{P_Z}\mathbf{P_W} - \mathbf{P_W}\mathbf{P_Z} + \mathbf{P_W} \\
&= \mathbf{P_Z} - \mathbf{P_Z}\mathbf{P_W} - (\mathbf{P_Z}\mathbf{P_W})^T + \mathbf{P_W} \\
&= \mathbf{P_Z} - 2\mathbf{P_W} + \mathbf{P_W} \\
&= \mathbf{P_Z} - \mathbf{P_W}
\end{aligned}
$$

- Note: we are using $\mathbf{P_Z}\mathbf{P_W} = \mathbf{P_W}$ as each column of $\mathbf{P_W}$ is in $C(\mathbf{W})$ and hence also in $C(\mathbf{Z})$

So $\mathbf{P_Z} - \mathbf{P_W}$ is a projection matrix

Projection Matrix $\mathbf{P_Z} - \mathbf{P_W}$

Onto what space is it projecting?

- Intuitively, it is projecting onto the part of $\mathbf{X}$ that is not in $\mathbf{W}$, $\tilde{\mathbf{X}} = (\mathbf{I} - \mathbf{P_W})\mathbf{X}$ (the part of $\mathbf{X}$ that is orthogonal to $\mathbf{W}$)
- $C(\tilde{\mathbf{X}})$ and $C(\mathbf{W})$ are complementary orthogonal subspaces of $C(\mathbf{Z})$
- $\mathbf{P_Z} - \mathbf{P_W}$ is a projection matrix onto $C(\tilde{\mathbf{X}})$ along $C(\mathbf{W})$
- we are decomposing $C(\mathbf{Z})$ into two orthogonal subspaces $C(\mathbf{W})$ and $C(\tilde{\mathbf{X}})$
- We can write $\mathbf{P_Z} = \mathbf{P}_{\tilde{\mathbf{X}}} + \mathbf{P_W}$ where $\mathbf{P}_{\tilde{\mathbf{X}}}\mathbf{P_W} = \mathbf{P_W}\mathbf{P}_{\tilde{\mathbf{X}}} = \mathbf{0}$

Note: we can always write

$$
\begin{aligned}
\boldsymbol{\mu} &= \mathbf{W}\boldsymbol{\alpha} + \mathbf{X}\boldsymbol{\beta} \\
&= \mathbf{W}\boldsymbol{\alpha} + (\mathbf{I} - \mathbf{P_W})\mathbf{X}\boldsymbol{\beta} + \mathbf{P_W}\mathbf{X}\boldsymbol{\beta} \\
&= \mathbf{W}\tilde{\boldsymbol{\alpha}} + \tilde{\mathbf{X}}\boldsymbol{\beta}
\end{aligned}
$$

Distribution of Extra Sum of Squares

- Since $\mathbf{P_Z} - \mathbf{P_W}$ is a projection matrix
- $\mathbf{Y}^T(\mathbf{P_Z} - \mathbf{P_W})\mathbf{Y}/\sigma^2$ has a χ^2_p distribution under M0

$$\begin{aligned}
\mathbf{Y}^T(\mathbf{P_Z} - \mathbf{P_W})\mathbf{Y} &= \|(\mathbf{P_Z} - \mathbf{P_W})\mathbf{Y}\|^2 \\
&= \|(\mathbf{P_Z} - \mathbf{P_W})(\mathbf{X}\boldsymbol{\beta} + \mathbf{W}\alpha + \boldsymbol{\epsilon}\|^2 \\
&= \|(\mathbf{P_Z} - \mathbf{P_W})(\mathbf{X}\boldsymbol{\beta}\boldsymbol{\epsilon}\|^2 \\
&= \|(\mathbf{P_Z} - \mathbf{P_W})\boldsymbol{\epsilon}\|^2 \quad \text{if } \boldsymbol{\beta} = \mathbf{0} \\
&= \boldsymbol{\epsilon}^T(\mathbf{P_Z} - \mathbf{P_W})\boldsymbol{\epsilon} \\
&\sim \sigma^2\chi^2_p \quad \text{if } \boldsymbol{\beta} = \mathbf{0}
\end{aligned}$$

- show that $\mathbf{Y}^T(\mathbf{P_Z} - \mathbf{P_W})\mathbf{Y}$ and $\mathbf{Y}^T(\mathbf{I} - \mathbf{P_Z})\mathbf{Y}$ are independent

F-Statistic

Under M1: $\boldsymbol{\beta} = \mathbf{0}$

$$\begin{aligned}
F(\mathbf{Y}) &= \frac{(\textsf{SSE}_{M0} - \textsf{SSE}_{M1})/p}{\textsf{SSE}_{M1}/(n-q-p)} \\
&= \frac{(\textsf{SSE}_{M0} - \textsf{SSE}_{M1})/\sigma^2 p}{\textsf{SSE}_{M1}/\sigma^2(n-q-p)} \\
&\stackrel{\text{D}}{=} \frac{\chi^2_p/p}{\chi^2_{n-q-p}/(n-q-p)} \\
&\stackrel{\text{D}}{=} F_{p,n-q-p}
\end{aligned}$$

Testing Individual Coefficients

Consider the model with $p = 1$, $\mathbf{Y} = \mathbf{W}\boldsymbol{\alpha} + \mathbf{x}\beta + \boldsymbol{\epsilon}$ and we want to test that $\beta = 0$ (M0)

1. fit the full model and compute $\textsf{SSE}_{M1}$
2. fit the reduced model and compute $\textsf{SSE}_{M0}$
3. calculate the F statistic and p-value

It turns out that we can obtain this F statistic by fitting the full model and the test reduces to a familiar t-test

$$
\begin{aligned}
\text{Note:} \qquad\qquad \textsf{SSE}_{M0} - \textsf{SSE}_{M1} &= \mathbf{Y}^T(\mathbf{P_Z} - \mathbf{P_W})\mathbf{Y} \\
&= \|(\mathbf{P}_{\tilde{\mathbf{X}}} + \mathbf{P_W} - \mathbf{P_W})\mathbf{Y}\|^2 \\
&= \|\mathbf{P}_{\tilde{\mathbf{X}}}\mathbf{Y}\|^2 \\
&= \|(\mathbf{I} - \mathbf{P_W})\mathbf{X}\hat{\boldsymbol{\beta}}\|^2 \\
&= \hat{\boldsymbol{\beta}}^T\mathbf{X}^T(\mathbf{I} - \mathbf{P_W})\mathbf{X}\hat{\boldsymbol{\beta}}
\end{aligned}
$$

Testing Individual Coefficients

For $p = 1$, the F statistic

$$
\begin{aligned}
F(\mathbf{Y}) &= \frac{(\mathsf{SSE}_{M0} - \mathsf{SSE}_{M1})/1}{\mathsf{SSE}_{M1}/(n-q-1)} \\
&= \frac{\hat{\beta}^T \mathbf{x}^T (\mathbf{I} - \mathbf{P_W}) \mathbf{x} \hat{\beta}}{s^2} \\
&= \frac{\hat{\beta}^2}{s^2 / \mathbf{x}^T (\mathbf{I} - \mathbf{P_W}) \mathbf{x}} \\
F(\mathbf{Y}) &\sim F_{1, n-q-1} \quad \text{under } \beta = 0
\end{aligned}
$$

- variance of $\hat{\beta}$:

$$
\begin{aligned}
\mathsf{var}[\hat{\beta}] &= \sigma^2 / \mathbf{x}^T (\mathbf{I} - \mathbf{P_W}) \mathbf{x} = \sigma^2 v \\
v &= 1 / \mathbf{x}^T (\mathbf{I} - \mathbf{P_W}) \mathbf{x}
\end{aligned}
$$

t-statistic

$$F(\mathbf{Y}) = \frac{\hat{\beta}^2}{s^2/\mathbf{x}^T(\mathbf{I}-\mathbf{P}_{\mathbf{W}})\mathbf{x}} = \left(\frac{\hat{\beta}}{s\sqrt{v}}\right)^2 = t(\mathbf{Y})^2$$

- Since $F(\mathbf{Y}) \sim F(1, n-q-1)$ under M0: $\beta = 0$, $t(\mathbf{Y})^2 \sim F(1, n-q-1)$ under M0: $\beta = 0$
- what is distribution of $t(\mathbf{Y})$ under M0: $\beta \neq 0$?

Recall that under M0: $\beta = 0$,

1. $\hat{\beta}/\sqrt{v\sigma^2} \sim \mathsf{N}(0,1)$
2. $(n-q-1)s^2/\sigma^2 \sim \chi^2_{n-q-1}$
3. $\hat{\beta}$ and s^2 are independent

Student t Distribution

▼ Theorem: Student t Distribution

A random variable T has a Student t distribution with ν degrees of freedom if

$$T \stackrel{\mathrm{D}}{=} \frac{Z}{X/\nu}$$

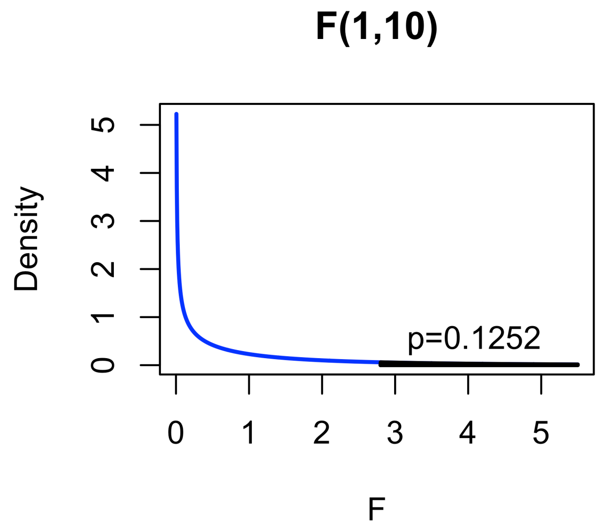
where

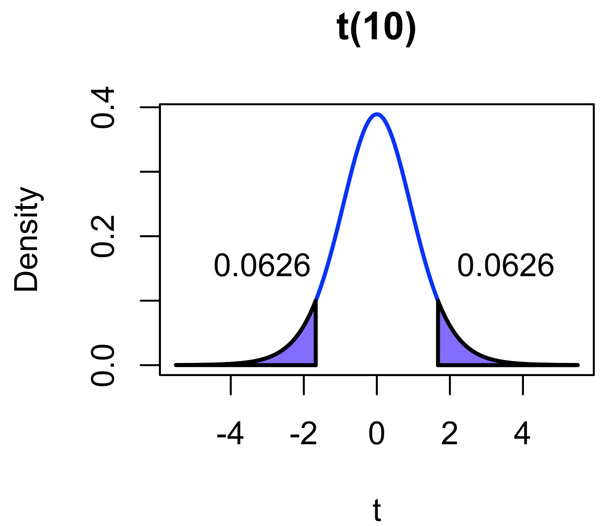
$$\begin{aligned}
&Z \sim \mathsf{N}(0,1) \\
&X \sim \chi^2_\nu \\
&Z \text{ and } X \text{ are independent}
\end{aligned}$$

- $\therefore\ t(\mathbf{Y}) = \hat{\beta}/\sqrt{v\sigma^2}$ has a Student t distribution with $n-q-1$ degrees of freedom under M0: $\beta = 0$

Decision rules and p-values

- an $F_{1,\nu}$ is equal in distribution to the square of Student t_ν distribution under the null model (also equal in distribution under the full model, but have a non-centrality parameter)
- Decision rule was to reject M0 if $F(\mathbf{Y}) > F_{1,n-q-1,\alpha}$
- p-value is $\Pr(F_{1,n-q-1} > F(\mathbf{Y})$; the probability of observing a value of F as extreme as the observed value under the null model
- using a t-distribution, the equivalent decision rule is to reject M0 if $|t(\mathbf{Y})| > t_{n-q-1,\alpha/2}$
- p-value is $\Pr(|T_{n-q-1}| > |t(\mathbf{Y})|)$
- equal-tailed t-test





Likelihood Ratio Tests

- we derived the F-test heurestically, but the formally this test may be derived as a likelihood ratio test.
- consider a statistical model $\mathbf{Y} \sim P, P \in \{P_{\boldsymbol{\theta}} : \boldsymbol{\theta} \in \boldsymbol{\Theta}\}$
- P is the true unknown distribution for $\mathbf{Y}$
- $\{P_{\boldsymbol{\theta}} : \boldsymbol{\theta} \in \boldsymbol{\Theta}\}$ is the model, the set of possible distributions for $\mathbf{Y}$ with $\boldsymbol{\Theta}$ the parameter space
- we might hypothesize that $\boldsymbol{\theta} \subset \boldsymbol{\Theta}_0 \subset \boldsymbol{\Theta}$
- for our linear model this translates as $\boldsymbol{\theta} = (\boldsymbol{\alpha}, \boldsymbol{\beta}, \sigma^2) \subset \mathbb{R}^q \times \{\mathbf{0}\} \times \mathbb{R}^+ \subset \mathbb{R}^g \times \mathbb{R}^p \times \mathbb{R}^+$
- compute the **likelihood ratio statistic**

$$R(\mathbf{Y}) = \frac{\sup_{\boldsymbol{\theta} \in \boldsymbol{\Theta}_0} p_{\boldsymbol{\theta}}(\mathbf{Y}))}{\sup_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} p_{\boldsymbol{\theta}}(\mathbf{Y}))}$$

Likelihood Ratio Tests

Equivalently, we can look at **-2 times the log likelihood ratio statistic**

$$\lambda(\mathbf{Y}) = -2\log(R(\mathbf{Y})) = -2[\sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}_0} l(\boldsymbol{\theta}) - \sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} l(\boldsymbol{\theta})]$$

where $l(\boldsymbol{\theta}) \propto \log p_{\boldsymbol{\theta}}(\mathbf{Y})$ (the log likelihood)

Steps:

1. Find the MLEs of $\boldsymbol{\theta}$ in the reduced model $\boldsymbol{\Theta}_0$, $\hat{\boldsymbol{\theta}}_0$
2. Find the MLEs of $the full model $\boldsymbol{\Theta}$, $\hat{\boldsymbol{\theta}}$
3. Compute $\lambda(\mathbf{Y}) = -2[l(\hat{\boldsymbol{\theta}}_0) - l(\hat{\boldsymbol{\theta}})]$
4. Find the distribution of $\lambda(\mathbf{Y})$ under the reduced model

with some rearranging and 1-to-1 transformations, can show that this is equivalent to the F-test! (HW)